

# Eigenvalues and extremal degrees in graphs

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## Abstract

Let  $G$  be a graph with  $n$  vertices,  $\mu_1(G) \geq \dots \geq \mu_n(G)$  be the eigenvalues of its adjacency matrix, and  $0 = \lambda_1(G) \leq \dots \leq \lambda_n(G)$  be the eigenvalues of its Laplacian. We show that

$$\delta(G) \leq \mu_k(G) + \lambda_k(G) \leq \Delta(G) \quad \text{for all } 1 \leq k \leq n,$$

and

$$\mu_k(G) + \mu_{n-k+2}(\overline{G}) \geq \delta(G) - \Delta(G) - 1 \quad \text{for all } 2 \leq k \leq n.$$

Let  $\mathcal{G}$  be an infinite family of graphs. We prove that  $\mathcal{G}$  is quasi-random if and only if  $\mu_n(G) + \mu_n(\overline{G}) = o(n)$  for every  $G \in \mathcal{G}$  of order  $n$ . This also implies that if  $\lambda_n(G) + \lambda_n(\overline{G}) = n + o(n)$  for every  $G \in \mathcal{G}$  of order  $n$ , then  $\mathcal{G}$  is quasi-random.

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## 1 Introduction

Our notation is standard (e.g., see [1], [2], and [5]); in particular, all graphs are defined on the vertex set  $\{1, 2, \dots, n\}$ ,  $G(n)$  stands for a graph of order  $n$ , and  $\overline{G}$  denotes the complement of  $G$ . Writing  $A(G)$  for the adjacency matrix of  $G$  and  $D(G)$  for the diagonal matrix of its degree sequence, the Laplacian of  $G$  is defined as  $L(G) = D(G) - A(G)$ . If  $G = G(n)$ , we order the eigenvalues of  $A(G)$  as  $\mu_1(G) \geq \dots \geq \mu_n(G)$  and the eigenvalues of  $L(G)$  as  $0 = \lambda_1(G) \leq \dots \leq \lambda_n(G)$ .

In this note we prove that if  $G = G(n)$  is a graph with minimum degree  $\delta(G)$  and maximum degree  $\Delta(G)$ , then

$$\delta(G) \leq \mu_k(G) + \lambda_k(G) \leq \Delta(G) \quad \text{for all } 1 \leq k \leq n. \quad (1)$$

This, in turn, implies that

$$\mu_k(G) + \mu_{n-k+2}(\overline{G}) \geq \delta(G) - \Delta(G) - 1 \quad \text{for all } 2 \leq k \leq n, \quad (2)$$

complementing the well-known inequality  $\mu_k(G) + \mu_{n-k+2}(\overline{G}) \leq -1$ .

In the second part of this note we give new spectral conditions for quasi-randomness of graphs. Throughout this note we denote by  $\mathcal{G}$  an infinite family of graphs. Following Chung, Graham, and Wilson [3], we call a family  $\mathcal{G}$  *quasi-random*, if for every  $G \in \mathcal{G}$  of order  $n$ ,

$$\mu_1(G) = 2e(G)/n + o(n), \quad \mu_2(G) = o(n), \quad \text{and} \quad \mu_n(G) = o(n).$$

Applying results of [6], we first prove the following theorem.

**Theorem 1** *A family  $\mathcal{G}$  is quasi-random if and only if*

$$\mu_n(G) + \mu_n(\overline{G}) = o(n) \tag{3}$$

*for every graph  $G \in \mathcal{G}$  of order  $n$ .*

This, in turn, implies the following sufficient conditions for quasi-randomness in terms of Laplacian eigenvalues.

**Theorem 2** *If  $\mathcal{G}$  is a family such that*

$$\lambda_n(G) + \lambda_n(\overline{G}) = n + o(n) \tag{4}$$

*for every  $G \in \mathcal{G}$  of order  $n$ , then  $\mathcal{G}$  is quasi-random.*

Since  $\lambda_2(G) + \lambda_n(\overline{G}) = n$  for every  $G = G(n)$ , we also obtain the following theorem.

**Theorem 3** *If  $\mathcal{G}$  is a family such that*

$$\lambda_2(G) + \lambda_2(\overline{G}) = o(n)$$

*for every  $G \in \mathcal{G}$  of order  $n$ , then  $\mathcal{G}$  is quasi-random.*

We leave the extension of the above results to normalized Laplacians to the interested reader.

## 2 Proofs

**Proof of inequality (1)** Let  $\mathbf{u}_1, \dots, \mathbf{u}_n$  be orthogonal unit eigenvectors to  $\lambda_1, \dots, \lambda_n$ . For every  $k = 2, \dots, n$ , the variational characterization of eigenvalues of Hermitian matrices ([5], p. 178-179) implies that

$$\lambda_k(G) = \min_{\|\mathbf{x}\|=1, \mathbf{x} \perp \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}\}} \langle L\mathbf{x}, \mathbf{x} \rangle \tag{5}$$

$$\mu_k(G) = \min_{M \subset \mathbb{R}^n, \dim M = k-1} \left\{ \max_{\|\mathbf{x}\|=1, \mathbf{x} \perp M} \langle A\mathbf{x}, \mathbf{x} \rangle \right\} \tag{6}$$

Let  $\mathbf{y}$  be such that  $\langle A\mathbf{y}, \mathbf{y} \rangle$  is maximal subject to  $\|\mathbf{y}\| = 1$  and  $\mathbf{y} \perp \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}\}$ . Letting  $\mathbf{y} = (y_1, \dots, y_n)$ , we find that

$$\begin{aligned} \lambda_k(G) &\leq \langle L\mathbf{y}, \mathbf{y} \rangle = \sum_{u \in V(G)} d(u) y_u^2 - \langle A\mathbf{y}, \mathbf{y} \rangle \leq \Delta(G) - \max_{\|\mathbf{x}\|=1, \mathbf{x} \perp \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}\}} \langle A\mathbf{x}, \mathbf{x} \rangle \\ &\leq \Delta(G) - \min_{M \subset \mathbb{R}^n, \dim M = k-1} \left\{ \max_{\|\mathbf{x}\|=1, \mathbf{x} \perp M} \langle A\mathbf{x}, \mathbf{x} \rangle \right\} = \Delta(G) - \mu_k(G), \end{aligned}$$

proving the second inequality of (1). The first inequality is deduced likewise using the dual version of (5) and (6).  $\square$

**Proof of inequality (2)** It is known that  $\lambda_k(G) + \lambda_{n-k+2}(\overline{G}) = n$  for all  $2 \leq k \leq n$ . This, in view of (1), implies that

$$\begin{aligned} n + \mu_k(G) + \mu_{n-k+2}(\overline{G}) &= \lambda_k(G) + \lambda_{n-k+2}(\overline{G}) + \mu_k(G) + \mu_{n-k+2}(\overline{G}) \\ &\geq \delta(G) + \delta(\overline{G}) \geq \delta(G) + n - 1 - \Delta(G), \end{aligned}$$

completing the proof of (2).  $\square$

**Proof of Theorem 1** The necessity of condition (3) is a routine fact, so we shall prove only its sufficiency. Let  $G = G(n)$ ,  $e(G) = m$ , and set  $s(G) = \sum_{u \in V(G)} |d(u) - 2m/n|$ . The following results were obtained in [6]

$$\frac{s^2(G)}{2n^2\sqrt{2m}} \leq \mu_1(G) - 2m/n \leq \sqrt{s(G)}, \quad (7)$$

$$\mu_k(G) + \mu_{n-k+2}(\overline{G}) \geq -1 - 2\sqrt{2s(G)} \quad \text{for all } 2 \leq k \leq n, \quad (8)$$

$$\mu_n(G) + \mu_n(\overline{G}) \leq -1 - s^2(G) / (2n^3). \quad (9)$$

Hence, if (3) holds, (9) implies  $\mu_n(G) = o(n)$ ,  $\mu_n(\overline{G}) = o(n)$ , and  $s(G) = o(n^2)$ . Thus, from (7) we obtain  $\mu_1(G) = 2m/n + o(n)$ . Since  $\mu_2(G) + \mu_n(\overline{G}) \leq -1$ , inequality (8) implies that  $\mu_2(G) = o(n)$ , completing the proof.  $\square$

**Proof of Theorem 2** According to Grone and Merris [4],  $\lambda_k(G) \geq \Delta(G)$ . Thus, (4) implies

$$n - 1 + \Delta(G) - \delta(G) = \Delta(G) + \Delta(\overline{G}) \leq \lambda_n(G) + \lambda_n(\overline{G}) = n + o(n).$$

Hence,

$$\Delta(G) - \delta(G) = \Delta(\overline{G}) - \delta(\overline{G}) = o(n)$$

and (1) implies

$$\begin{aligned} \mu_n(G) &= -\lambda_n(G) + \Delta(G) + o(n) \\ \mu_n(\overline{G}) &= -\lambda_n(\overline{G}) + \delta(\overline{G}) + o(n). \end{aligned}$$

Adding these two inequalities, in view of (4), we obtain  $\mu_n(G) + \mu_n(\overline{G}) = o(n)$ ; the assertion follows from Theorem 1.  $\square$

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